Replica model at low integer N for directed polymers in $(1+1)$ dimensions

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We study directed polymers in a $(1+1)$-dimensional disordered environment with discrete space and time. For fixed L (polymer length) and r (disorder parameter) the quantity $\langle Z^n \rangle$ has two crossovers in n, one related to the radius of convergence of the cumulant expansion for $\ln Z$ and one related to the discreteness of the lattice. The existence of the first crossover [at $n_1(L) \to 0$ as $L \to \infty$] weakens the customary argument relating the behavior of the cumulants of $\ln Z$ to the n dependence of $E_n \equiv -\lim_{L\to\infty} \ln\langle Z^n \rangle / L$. The second crossover [at $n_2(r) \to \infty$ as $r \to 0$] is explored here by computing E_2 analytically and E_3 numerically to high accuracy for several values of r in two models.

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I. INTRODUCTION

The problem of directed polymers in random media has been receiving much attention [1–4]. Besides its direct connection to the randomized Ising lattice [1], it is related by a mapping via the stochastic Burgers equation [5] to many other physical phenomena: the growth of an interface in the Eden model [6], ballistic aggregation [7], domain walls in the two-dimensional random bond Ising model [8], and a randomly stirred fluid [9] obeying Burgers equation [10].

It seems that the "replica method" is one of the most popular tools in dealing with the directed polymer problem. The "replicated partition function" can be converted into the multiple random walk with attraction (MRWA) which may be defined as follows: n particles execute *simultaneous* random walks on a lattice in discrete time. One considers the ensemble of n-fold walks of L steps, with each n-fold walk weighted by an enhancement factor $Y[x_1, \ldots, x_n]$ exponentially related to the degree of coincidence between particles. (We shall write x_i for the whole sequence $x_i(t)$ of positions of the ith particle at $t = 1, 2, \ldots, L$.) We may distinguish the following two versions:

$$Y = \prod_{i,j} (1+r)^{s_{ij}} \quad \text{(static attraction)}, \tag{1.1}$$

where r is a given parameter >0 ($r=0$ yields the trivial "free-particle" case) and s_{ij} is the number of integers t ($0 \le t < L$) for which $x_i(t) = x_j(t)$, i.e., the length of time during which the particles coincide; and

$$Y = \prod_{t=1}^{T} \prod_{x} S(x,t) \quad \text{(combinatoric attraction)}, \tag{1.2}$$

where $S(x,t)$ depends on the number n_{xt} of particles located at x after $t-1$ steps and on the partition of n_{xt} effected by the tth step.

Kardar [1] considered a two-dimensional Ising lattice with random bond strengths and a domain wall imposed by external boundary conditions. In the solid-on-solid approximation, which makes the interface into a directed polymer, he showed that if the bond strengths are independently distributed by a single Gaussian then the average "n-fold replicated" partition function $\langle Z^n \rangle$ (where $\langle\ \rangle$ denotes disorder average) is equivalent to the partition function $\sum_{x_1,\ldots,x_n} Y$ of an n-body static MRWA in one dimension.

Medina and Kardar [2] considered a directed polymer system in which the bond strengths are all equal but each bond contributes a random sign to any polymer traversing it. They found that $\langle Z^{2n} \rangle$ was given by an n-body combinatoric MRWA. The idea is that the polymers must travel in paired strands to prevent vanishing of the average over signs; when two of the n pairs come together and then separate, there are three ways for the pairs to reform. In general, if n_{xt} pairs occupy site x at time $t-1$ and these pairs separate into sets of $n_{xt1}, n_{xt2}, \ldots$, at time t, where $\sum_j n_{xtj} = n_{xt}$, then the enhancement factor in (1.2) is

$$S(x,t) = \frac{(2n_{xt})!}{\prod_j (2n_{xtj})!} \Big/ \frac{n_{xt}!}{\prod_j n_{xtj}!} \ . \tag{1.3}$$

The standard approach [1,2,11] to the MRWA has been to replace it by a quantum statistical continuum model in which n particles obey a Schrödinger equation with Hamiltonian

$$H = \sum_{j=1}^{n} -\frac{1}{2} \left[\frac{\partial}{\partial x_j} \right]^2 - \lambda \sum_{i,j}^{n} 2\delta(x_i - x_j) \ , \tag{1.4}$$

where $\lambda = \ln(1+r)$. The factor 2 in the second term arises because in our lattice x varies by steps of two units at a fixed t.

Since the ground-state energy of this $\hat{H}$ in one dimen-

sion is $-(n^3-n)\lambda^2/6$, it has been claimed that the (1+1)-dimensional MRWA (both static and combinatoric) has a partition function

$$Z_n(L)=\sum_{x_1,\ldots,x_n} Y=e^{-F_n(L)}Z_1(L)^n \tag{1.5}$$

where

$$F_n(L)\underset{L\to\infty}{\simeq} L(n-n^3)\lambda^2/6\ . \tag{1.6}$$

From the form of this expression, particularly the absence of n^2 and the presence of n^3, various inferences have been drawn about the critical exponents in the directed polymer problem. In particular the coefficients $C_m(L)$ of the power series $F_n(L)=\sum_m C_m n^m/m!$ can be identified [1] with the statistically observable cumulants of the distribution of $\Delta\ln Z\equiv\ln Z-\langle\ln Z\rangle$ in the randomized directed polymer problem. From (1.6) one deduces that C_2, the variance, grows more slowly than L but that C_3 grows as L; this implies that $\ln Z$ has a *skew* distribution with width $\sim L^{\nu}$ where $\frac{1}{3}\le\nu<\frac{1}{2}$. Numerical simulations [3,4] give $\nu\simeq\frac{1}{3}$.

The argument sketched above has two points of vulnerability. The first one has to do with the connection between $C_m(L)$ and $\lim_{L\to\infty}F_n(L)/L$; the second, with the validity of the continuum approximation for the replica problem on the lattice.

The first point is given prominence by a recent numerical simulation [4] of the directed polymer showing that C_4 grows as $L^{4/3}$. Naively this would seem to require a term $\sim n^4L^{4/3}$ in $F_n(L)$, which not only contradicts (1.6) but seems impossible since it is easily shown in the static attraction model that

$$\begin{aligned}Z_n(L)&\equiv\sum_{x_1,\ldots,x_n} Y(x_1,\ldots,x_n,t)\\ &\le h^n(1+r)^{n(n-1)/2}Z_n(L-1)\ ,\end{aligned} \tag{1.7}$$

where h is the branching number at each step ($=2$ in our walk); and consequently that $\lim_{L\to\infty}F_n(L)/L$ is finite. Moreover, it is tempting to extend the result of [4] to a scaling conjecture, that the distribution of $(\ln Z-\langle\ln Z\rangle)/L^{1/3}$ approaches a fixed form at large L; the conjecture would imply $C_m\sim L^{m/3}$. This seems to violate (1.7) even more strongly.

A more careful examination of the possible forms of functions of two variables shows, however, that the result of [4] does not require that $F_n(L)$ grow faster than L for fixed n. In fact, we can make the stronger statement: *even the specific form (1.6) is compatible not only with the scaling of C_4 as $L^{4/3}$ but with the full scaling hypothesis $C_m\sim L^{m/3}$ for all $m>1$.*

We establish this fact by means of a "toy" form of $F_n(L)$: suppose that

$$F_n(L)=[nL-w(nL^{1/3})]\lambda^2/6\ , \tag{1.8}$$

where $w(x)$ is some analytic function which behaves as x^3 for large positive x. (For example, $w(x)=x^4/(x+b)$ or $w(x)=x^3+e^{-ax}$.) For any fixed n, as $L\to\infty$ we have $w(nL^{1/3})\sim n^3L$ reproducing the continuum result (1.6). But at fixed L the Taylor expansion $w(x)=\sum_m b_m x^m/m!$ yields $C_m=-\lambda^2 b_m L^{m/3}/6$ in agreement with the scaling hypothesis [12].

Thus, the behavior of the cumulants for large L is reconciled with that of $F_n(L)$, but at the cost of postulating a crossover between the regime $n\to 0$ for fixed L and the regime $L\to\infty$ for fixed n. In view of this separation of regimes, the original reasoning from (1.6) to the conclusion that $\lim_{L\to\infty}C_2/L=0$ becomes quite weak. The most we can say is that *if* the distribution of $\ln Z$ about its mean is assumed to have some shape independent of L with a width $\delta(L)$, so that the distribution is a function of $[\ln Z-\langle\ln Z\rangle]/\delta(L)$, then $C_m\sim\delta^m$ and $F_n(L)$ is a function of $n\delta(L)$ alone, at least when $n\delta(L)$ is small; and *if* also $F_n(L)$ can be analytically continued between the two regimes, then it must also depend only on $n\delta(L)$ as $L\to\infty$. Only then can we infer from (1.6) that $\delta(L)$ must scale as $L^{1/3}$, and, hence, $C_m\sim L^{m/3}$.

Even if the two supplementary assumptions [scaling of $(\ln Z-\langle\ln Z\rangle)$ and analytic continuation] are made, one cannot deduce the coefficient $\lim_{L\to\infty}C_3/L$ from (1.6), because there is no need for the Taylor expansion of $w(x)$ in (1.8) to have $b_3=1$ [see the second example after (1.8)]. In fact, a recent simulation [13] gives $\lim_{L\to\infty}C_3/L=-0.145$ for a certain choice of parameters (Figs. 9 and 11, therein) whereas by assuming $b_3=1$ one would have predicted $\lim_{L\to\infty}C_3/L=-\theta/4=-1.71/4\sim-0.43$ (see Table II therein).

In the absence of these supplementary assumptions the lack of connection between (1.6) and the behavior of C_m is illustrated by various alternatives to (1.8). For example, if we add to $F_n(L)$ a term $u(nL^{1/2})$ where $u(x)\sim x^2$ as $x\to 0$ but $u(x)\sim x$ as $x\to\infty$ [e.g., $u(x)=\sqrt{1+x^2}-1$] then (1.6) is unaffected because the new term grows only as $L^{1/2}$ for fixed n; nevertheless C_2 now grows linearly with L. Or again, let us add to (1.8) a term $v(nL^{1/4})$ where $v(x)\sim x^4$ for large x. This term will change (1.6) by making it quartic in n. But it will have no effect on the leading behavior of the cumulants for large L, since $L^{m/4}$ is negligible in comparison with $L^{m/3}$.

It is true that a term like n^4L is ruled out by direct analysis of the replica model and that numerical simulations show that such terms as $\sqrt{1+n^2L}-1$ are not present. But these same simulations have already told us directly $C_2\sim L^{2/3}$. It therefore seems to us illogical to represent the behavior of C_2 as a deduction from (1.6). More appropriately, the observed behavior of the cumulants can be *combined* with the result (1.6) to imply that there is probably an analytic continuation between the two regimes, since the combination $nL^{1/3}$ is important in both.

The second point is brought out by observing that for a fixed r, (1.6) cannot possibly hold for all n [14] because (1.7) shows that $[\lim_{L\to\infty}F_n(L)/L]$ is bounded above by $[n(n-1)\ln\sqrt{1+r}+O(n)]$. Therefore, for fixed L and r, there are two crossovers and three regimes. Let us call these three the following.

(a) The *cumulant* regime: $n\le n_1(L,r)$ where $F_n(L)$ is given accurately by the series $\sum_m C_m[n]^{m/m}$.

(b) The *Bethe* regime: $n_1(L,r)\le n\le n_2(L,r)$ in which

(1.6) holds.

(c) The *discrete* regime: $n \geq n_2(L,r)$ in which the replica model yields probability functions that change drastically over one lattice step. Here, $F_n(L)$ can be approximated by assuming that (1.7) is saturated.

The nature of the two crossovers can be borne in mind by recalling that $n_1 \to 0$ for large L, whereas $n_2 \to \infty$ for small r. The second crossover has been discussed by Medina and Kardar [14], who argue plausibly that it has no effect on the continuation of the cumulant regime from small to larger r. It seems worthwhile nevertheless to improve our grasp of the entire situation by studying this crossover through a calculation of $E_n \equiv \lim_{L\to\infty} F_n(L)/L$ for small integer n over a wide range of values of r. Setting

$$E_n = \ln(2^n q_n) \tag{1.9}$$

we note that the generating function G_q of the MRWA has a singularity at $q = q_n$. This singularity can be located analytically for $n = 2$ and numerically for $n = 3$, and so we can obtain E_n as a function of r. In the next two sections we describe this calculation, and in the last section we discuss the second crossover in the light of it.

II. ANALYTICAL SOLUTION FOR $n = 2$

Starting from some fixed initial condition, we write $\psi(x_1,x_2,t)$ for the weighted sum of all two-fold walks bringing the two particles to x_1 and x_2 after t steps. The function ψ is analogous to an unnormalized two-body Schrödinger function. It obeys the recursion

$$\psi(x_1,x_2,t) = \sum_{j_1=\pm1} \sum_{j_2=\pm1} \psi(x_1-j_1,x_2-j_2,t-1) \times [1 + r\Delta(x_1,x_2,j_1,j_2)] , \tag{2.1}$$

where the function Δ distinguishes the two versions:

$$\Delta = \begin{cases} \delta_{x_1-j_1,x_2-j_2} & \text{(static attraction)} \\ \delta_{x_1-j_1,x_2-j_2}(1-\delta_{x_1,x_2}) & \text{(combinatoric attraction).} \end{cases} \tag{2.2}$$

It is clear that $\psi(x_1,x_2,t)$ records all the enhancement up to step $t-1$. For definiteness we shall suppose that two particles begin together at the origin

$$\psi(x_1,x_2,0) = \delta_{x_1,0}\delta_{x_2,0} . \tag{2.3}$$

It is useful to introduce the "free-particle" Green's function which satisfies (2.1), with $r = 0$, and (2.3),

$$G(x_1,x_2,t) = \sum_{j_1=\pm1} \sum_{j_2=\pm1} G(x_1-j_1,x_2-j_2,t-1) , \qquad G(x_1,x_2,0) = \delta_{x_1,0}\delta_{x_2,0} . \tag{2.4}$$

[Of course, G factors as $\binom{t}{(t+x_1)/2}\binom{t}{(t+x_2)/2}$, but we shall not find this form useful.]

The solution to (2.1) and (2.2) may be expressed in terms of G,

$$\psi(x_1,x_2,t) = G(x_1,x_2,t) + r \sum_{t'=1}^{t} \sum_{x_1',x_2'} \sum_{j_1,j_2=\pm1} G(x_1-x_1',x_2-x_2',t-t')\Delta(x_1',x_2',j_1,j_2)\psi(x_1'-j_1,x_2'-j_2,t'-1) . \tag{2.5}$$

We now introduce the generating functions

$$\psi_q(x_1,x_2) \equiv \sum_{t=0}^{\infty} q^t \psi(x_1,x_2,t) , \qquad G_q(x_1,x_2) \equiv \sum_{t=0}^{\infty} q^t G(x_1,x_2,t) . \tag{2.6}$$

These will, of course, converge only for sufficiently small q. If, indeed, the partition function $\sum Y = \sum_{x_1,x_2,\ldots,x_n} \psi(x_1,x_2,\ldots,x_n,t)$ behaves as $2^{nt} e^{-\hat{H}t}$ for large t, where $\hat{H}$ is the Hamiltonian in (1.4), then it is seen from (2.6) that $\sum_{x_1,x_2} \psi_q(x_1,x_2)$ should encounter its first singularity at $2^2 q = e^{E_2}$, where E_2 is the lowest eigenvalue of $\hat{H}$ for $n = 2$.

(2.5) now becomes

$$\psi_q(x_1,x_2) = G_q(x_1,x_2) + rq \sum_{x_1',x_2'} \sum_{j_1,j_2=\pm1} \psi_q(x_1'-j_1,x_2'-j_2) G_q(x_1-x_1',x_2-x_2')\Delta(x_1',x_2',j_1,j_2) . \tag{2.7}$$

It is now helpful to introduce the "relative coordinate" $y = (x_1-x_2)/2$ and the "total momentum $2k$". We define

$$\psi_q^k(y) \equiv \sum_{x_1,x_2} e^{-ik(x_1+x_2)} \delta_{x_1-x_2,2y} \psi_q(x_1,x_2) , \qquad G_q^k(q) \equiv \sum_{x_1,x_2} e^{-ik(x_1+x_2)} \delta_{x_1-x_2,2y} G_q x_1,x_2) . \tag{2.8}$$

We then have

$$\psi_q^k(y)=G_q^k(y)+rqJ_q^k(y)\psi_q^k(0)\ , \tag{2.9}$$

where

$$J_q^k(y)=\begin{cases}G_q^k(y+1)+G_q^k(y-1)+2\cos(2k)G_q^k(y) & \text{(static case)}\\ G_q^k(y+1)+G_q^k(y-1) & \text{(combinatoric case).}\end{cases} \tag{2.10}$$

Define

$$\begin{aligned}\psi_q^{kl}&\equiv\sum_y e^{-2ily}\psi_q^k(y)\ ,\\ G_q^{kl}&\equiv\sum_y e^{-2ily}G_q^k(y)\ ,\\ J_q^{kl}&\equiv\sum_y e^{-2ily}J_q^k(y)\ .\end{aligned} \tag{2.11}$$

It is straightforward to calculate G_q^{kl} by transforming (2.4) according to (2.6), (2.8), and (2.11),

$$G_q^{kl}=\frac{1}{1-2q\cos 2l-2q\cos 2k}\ . \tag{2.12}$$

Then we have

$$\psi_q^{kl}=G_q^{kl}+rqJ_q^{kl}\psi_q^k(0)\ , \tag{2.13}$$

where

$$J_q^{kl}=\begin{cases}2(\cos 2k+\cos 2l)G_q^{kl} & \text{(static case)}\\ 2\cos(2l)G_q^{kl} & \text{(combinatoric case).}\end{cases} \tag{2.14}$$

From (2.9) we have

$$\psi_q^k(0)=\frac{G_q^k(0)}{1-rqJ_q^k(0)}\ . \tag{2.15}$$

From the definitions of G_q^{kl}, J_q^{kl} we know that

$$\begin{aligned}G_q^{kl}(0)&=\frac{1}{2\pi}\int G_q^{kl}e^{il0}dl=\frac{1}{2\pi}\int G_q^{kl}dl\\ &=1/[(1+4q\sin^2k)(1-4q\cos^2k)]^{1/2}\ ,\end{aligned}$$

and by comparing (2.12) with (2.14) (2.16)

$$J_q^k(0)=\begin{cases}[G_q^k(0)-1]/q & \text{(static case)}\\ [(1-2q\cos 2k)G_q^k(0)-1]/q & \\ \text{(combinatoric case).}\end{cases}$$

Using the above equations (2.13)–(2.16), we find that

$$\psi_q^{kl}=\begin{cases}G_q^{kl}\left[1+\dfrac{2rq(\cos 2k+\cos 2l)G_q^k(0)}{1+r-rG_q^k(0)}\right]\\ \text{(static case)}\\ G_q^{kl}\left[1+\dfrac{2rq\cos 2lG_q^k(0)}{1+r-r(1-2q\cos 2k)G_q^k(0)}\right]\\ \text{(combinatoric case).}\end{cases} \tag{2.17}$$

Thus we have solved the two-particle problem analytically.

We are interested in the quantity $\sum_{x_1,x_2}\psi_q(x_1,x_2)$, which gives us the Laplace-transformed partition function and whose smallest pole with $2^2q<1$ will tell us the ground-state energy $(\ln 4q)$. It is obvious that the Laplace-transformed partition function is just $\psi_q^{k=0,l=0}$.

The pole we want is from the singularity of $\psi_q^k(0)$, i.e., $1+r=rG_q^{k=0}(0)$ for the static case and $1+r=r(1-2q)G_q^{k=0}(0)$ for the combinatoric case. We therefore find referring to (2.16) with $k=0$ that the poles are located at

$$\frac{1+r}{r}=\begin{cases}\dfrac{1}{(1-4q_2)^{1/2}} & \text{(static case)}\\ \dfrac{1-2q_2}{(1-4q_2)^{1/2}} & \text{(combinatoric case),}\end{cases} \tag{2.18}$$

where q_2 represents the critical q value for two particles. The energy $E_2(r)$ is obtained by solving (2.18) for q_2 and using the result in (1.9).

III. NUMERICAL SOLUTION FOR $n=3$

For more than two particles, one cannot solve the Laplace-transformed partition function analytically. However, from the experience of solving the two-particle case we can perform a numerical calculation to find out the pole of the Laplace-transformed partition function in the three-particle case.

Corresponding to (2.1) and (2.2) we have

$$\begin{aligned}\psi(x_1,x_2,x_3,t)=&\sum_{j_1,j_2,j_3=\pm1}\psi(x_1-j_1,x_2-j_2,x_3-j_3,t-1)\\ &\times[1+r\Delta(x_1,x_2,x_3,j_1,j_2,j_3)]\end{aligned} \tag{3.1}$$

with

$$\Delta=\sum_{(\alpha,\beta)=(1,2)(2,3)(3,1)}\delta_{x_\alpha-j_\alpha,x_\beta-j_\beta}f_{\alpha\beta}$$

where assuming $\alpha\neq\gamma$,

$$f_{\alpha\beta}=\begin{cases}1+\frac{1}{3}(r^2+3r)\delta_{x_\gamma-j_\gamma,x_\alpha-j_\alpha} & \text{(static case)}\\ 1-\delta_{x_\alpha,x_\beta} & \text{(combinatoric case).}\end{cases} \tag{3.2}$$

This choice of $f_{\alpha\beta}$ for the static case gives the desired enhancement $(1+r)^3$ when all three particles coincide. For the combinatoric case (3.1) is correctly reproduced. As before, $\psi(x_1,x_2,x_3,t)$ records all enhancement effects

up to step $t-1$. For the initial condition, we take

$$\psi(x_1,x_2,x_3,t=0)=\delta_{x_1,0}\delta_{x_2,0}\delta_{x_3,0}\ . \tag{3.3}$$

Again, we define G as the specialization of ψ when $r=0$, and the generating functions ψ_q, G_q in the manner of (2.6). We obtain, corresponding to (2.7),

$$\psi_q(\mathbf{x})\equiv\sum_{t=0}^{\infty}q^t\psi(\mathbf{x},t)$$
$$=G_q(\mathbf{x})+rq\sum_{\mathbf{x}'}G_q(\mathbf{x}-\mathbf{x}')\sum_{\mathbf{j}=(\pm1,\pm1,\pm1)}\Delta(\mathbf{x}',\mathbf{j})\psi_q(\mathbf{x}'-\mathbf{j})$$

or

$$\psi_q(x_1,x_2,x_3)=G_q(x_1,x_2,x_3)+rq\sum_{\bar{x},\bar{q}}\psi_q(\bar{x},\bar{x},\bar{x}-\bar{y})[1+s\tfrac{1}{3}(r^2+3r)\delta_{\bar{y},0}]$$
$$\times[J_q(x_1-\bar{x},x_2-\bar{x},x_3-\bar{x}+\bar{y})+J_q(x_2-\bar{x},x_3-\bar{x},x_1-\bar{x}+\bar{y})$$
$$+J_q(x_3-\bar{x},x_1-\bar{x},x_2-\bar{x}+\bar{y})]\ , \tag{3.4}$$

where

$$s=\begin{cases}0 & \text{for combinatoric case}\\ 1 & \text{for static case}\end{cases}$$

and

$$J_q(x_1,x_2,x_3)=\sum_{j_1,j_2,j_3=\pm1}G_q(x_1-j_1,x_2-j_2,x_3-j_3)\frac{s+1}{2}[1+(s-1)j_1j_2] \tag{3.5}$$

and

$$G_q(x_1,x_2,x_3)=\frac{1}{(2\pi)^3}\int d^3k e^{-ik_1x_1-ik_2-ik_3x_3}/(1-8q\cos k_1\cos k_2\cos k_3)\ .$$

We intend to introduce (redundant) relative coordinates

$$y_1=(x_2-x_3)/2,\quad y_2=(x_3-x_1)/2,\quad y_3=(x_1-x_2)/2 \tag{3.6}$$

with a constraint

$$y_1+y_2+y_3=0\ ; \tag{3.7}$$

because of (3.7) the y's define a plane, not a three space. But having defined

$$\psi_q^k(y_1,y_2,y_3)\equiv\sum_{x_1,x_2,x_3}e^{-ik(x_1+x_2+x_3)}\delta_{x_2-x_3,2y_1}\delta_{x_3-x_1,2y_2}\delta_{x_1-x_2,2y_3}\psi_q(\mathbf{x})\ ,$$
$$G_q^k(y_1,y_2,y_3)\equiv\sum_{x_1,x_2,x_3}e^{-ik(x_1+x_2+x_3)}\delta_{x_2-x_3,2y_1}\delta_{x_3-x_1,2y_2}\delta_{x_1-x_2,2y_3}G_q(\mathbf{x})\ ,$$
$$J_q^k(y_1,y_2,y_3)\equiv\sum_{x_1,x_2,x_3}e^{-ik(x_1+x_2+x_3)}\delta_{x_2-x_3,2y_1}\delta_{x_3-x_1,2y_2}\delta_{x_1-x_2,2y_3}J_q(\mathbf{x})\ , \tag{3.8}$$

we find that (3.4) transforms into an equation with three sources, arising from $y_1=0$, $y_2=0$, and $y_3=0$. Thus the functions $\psi_q^k(0,y,-y)$, $\psi_q^k(-y,0,y)$, and $\psi_q^k(y,-y,0)$ all influence one another. To be sure, these are all the same functions by symmetry; but the cross connections involve a kernel, e.g., $G_q^k[0-(-y'),y-0,-y-y']=G_q^k(y',y,-y-y')$, that is not translation independent with respect to the y variable. Hence, we cannot complete the solution analytically.

Our plan is, having written the analog to (2.9) and specialized to the case $y_3=0$, to solve it by iteration as a one-dimensional nonlocal diffusion process with a kernel determined by G. Although the diffusion process takes place on the line $y_3=0$, the kernel is obtained as a sum over free propagators on a planar triangular lattice. Fortunately we are in possession of a method [15] for obtaining these propagators quickly to high accuracy (see the Appendix).

By transforming (3.5) according to (3.8), we get

$$J_q^k(y_1,y_2,y_3)=2s\cos(3k)G_q^k(y_1,y_2,y_3)+e^{ik}[G_q^k(y_1+1,y_2,y_3-1)+G_q^k(y_1,y_2-1,y_3+1)+sG_q^k(y_1-1,y_2+1,y_3)]$$
$$+e^{-ik}[G_q^k(y_1-1,y_2,y_3+1)+G_q^k(y_1,y_2+1,y_3-1)+sG_q^k(y_1+1,y_2-1,y_3)] \tag{3.9}$$

and (3.4) now becomes

$$\psi_q^k(y_1,y_2,y_3)=G_q^k(y_1,y_2,y_3)+rq\sum_{\bar{y}}[1+s\tfrac{1}{3}(r^2+3r)\delta_{\bar{y},0}]\psi_q^k(\bar{y},-\bar{y},0)$$
$$\times[J_q^k(y_1-\bar{y},y_2+\bar{y},y_3)+J_q^k(y_2-\bar{y},y_3+\bar{y},y_1)+J_q^k(y_3-\bar{y},y_1+\bar{y},y_2)]\ ; \quad (3.10)$$

taking $y_3=0$, the above equation becomes

$$\psi_q^k(y)\equiv\psi_q^k(y,-y,0)$$
$$=G_q^k(y,-y,0)+r\sum_{y'}\psi_q^k(y')L_q^k(y,y')+srq(r^2+3r)\psi_q^k(0)J_q^k(y,-y,0)\ , \quad (3.11)$$

where

$$L_q^k(y,y')=q[J_q^k(y-y',-y+y',0)+J_q^k(-y,-y',y',y)+J_q^k(-y',y+y',-y)]\ . \quad (3.12)$$

[Note that only the first term in (3.12) is a function of $y-y'$ alone.]

From iterating (3.11) with $k=0$ we can find out the corresponding r for a given q_3 which is less than $\frac{1}{8}$. And we find that r increases as q_3 decreases which agrees with our intuition. (The stronger the attraction, the lower the ground-state energy.)

We will drop the subscript of q_3 for convenience in the following discussion. The way to find out r for a given q_3 is the following:

Note that L_q^k is composed of free propagators G_q^k, and $G_q^k(y_1,y_2,y_3)$ decays exponentially as it goes away from the origin if $q<\frac{1}{8}$. Therefore, we can control the accuracy of (3.11) by controlling the sum range y' in (3.11). (We continue to write the superscript k but it should be understood that we have set $k=0$.)

For simplicity, let's consider the combinatoric case first: We first give a trial solution $\psi_{q(1)}^k(y)$, iterate it once by the kernel $L_q^k(y,y')$ only, and call the normalized iterated result $\psi_{q(2)}^k(y)$, i.e.,

$$\psi_{q(2)}^k(y)=r_1\sum_{y'}\psi_{q(1)}^k(y')L_q^k(y,y')$$

where the normalization constant r_1 is chosen so that $\psi_{q(2)}(0)=\psi_{q(1)}(0)$. We then repeat the process successively,

$$\psi_{q(i+1)}^k(y)=r_i\sum_{y'}\psi_{q(i)}^k(y')L_q^k(y,y')\ . \quad (3.13)$$

We continue this iteration process until the solution is "stabilized," i.e., until $\psi_{q(i+1)}^k(y)\simeq\psi_{q(i)}^k(y)$ for all y.

For the static case, the iteration equation (3.13) is replaced by

$$\psi_{q(i+1)}^k(y)=r_i\left[\sum_{y'}\psi_{q(i)}^k(y')L_q^k(y,y')\right.$$
$$+q(r_{i-1}r_{i-2}+3r_{i-1})$$
$$\left.\times\psi_{q(i)}^k(0)J_q^k(y,-y,0)\right] \quad (3.14)$$

with the condition that we replace the subscript of r by 1 when the subscript is less than 1.

And now we set $r=\lim_{i\to\infty}r_i$; then the q we have been using is critical for this r, since if q is made any larger we get

$$\left|r\sum_{y'}\psi_q^k(y')L_q^k(y,y')+srq(r^2+3r)\psi_q^k(0)J_q^k(y,-y,0)\right|$$
$$>|\psi_q^k(y)|,\quad \text{for all } y$$

and the partition function in (3.11) diverges, but if q is smaller it converges.

IV. RESULTS AND CONJECTURES

From Sec. I, we know that if the continuum approximation to MRWA, represented by (1.6), is correct, we must have

TABLE I. Results for $n=2$ and $n=3$, static case. For each value of r (column 2) the "ground-state energy" E_2 (E_3) is the logarithm of the quantity in column 3 (column 1). The ordering of the columns reflect our method of calculation. Column 4 is the ratio of the two logarithms. Column 5 uses the quantity $E_3(\text{pred})=-24\lambda^2 g(3\lambda)/6$ where $\lambda=\ln(1+r)$ and g is defined by Eq. (4.4).

2^3q_c ($n=3$)	r	2^2q_c ($n=2$)	E_3/E_2	$E_3(\text{pred})/E_2$
0.050	3.189 566	0.420 404	3.457 126	3.359 652
0.150	1.809 702	0.585 148	3.540 123	3.439 058
0.300	1.136 768	0.716 972	3.618 597	3.514 734
0.450	0.789 914	0.805 242	3.686 344	3.578 731
0.650	0.488 682	0.892 241	3.778 209	3.665 067
0.800	0.310 828	0.943 725	3.856 426	3.742 545
0.950	0.128 051	0.987 114	3.954 942	3.864 025
0.965	0.104 428	0.991 060	3.967 112	3.884 835
0.980	0.076 613	0.994 936	3.979 455	3.911 470

$$\frac{E_3}{E_2}=\frac{\ln[2^3q_3]}{\ln[2^2q_2]}=\frac{3^2-3}{2^3-2}=4\ . \tag{4.1}$$

On the other hand, if we take $E_n\sim-\lambda n(n-1)/2$ as suggested by (1.7) for the static model, we get

$$E_3/E_2=3\times2/2\times1=3\ . \tag{4.2}$$

Hence in the static model we expect $E_3/E_2\simeq4$ when $n_2\gg3$, and $E_3/E_2\simeq3$ when $n_2\ll2$. In Table I we show E_3/E_2 for several values of r; the procedure was to choose q_3, compute r by the method of Sec. III, and derive the corresponding q_2 from (2.18). We see that $E_3/E_2\simeq3\frac{1}{2}$ for $r\simeq2\frac{1}{2}$, and therefore we can say roughly that n_2 lies between 2 and 3 when r is in this vicinity. For the combinatoric model, at large r, E_3 should be dominated by the detachment of one pair strand from the other two, for which (13) gives $Z_3(L)\simeq[Z_2(L)]^2$; hence, we expect $E_3/E_2\simeq2$ when $n_2\ll2$, and again $E_3/E_2\simeq4$ when $n_2\gg3$. We find (Table II) $E_3/E_2\simeq3$ when $r\simeq2$, and so we infer that n_2 lies between 2 and 3 for this r. (The value $r=2$ is just that which pertains to the application of [2].)

Regarding $F_n(L)$ as a function of three variables n, L, r, we have speculated about the existence of some simple form analogous to (1.6) which might extend across *both* crossovers. It is not possible to do this with a function of one variable, but perhaps the three variables can be reduced to two determining the part of F_n not linear in L. We reason as follows for the static case:

For very large r [or $\lambda=\ln(1+r)$] we know the particles will stick together to get maximum enhancement and the "growing rate" (or the "ground-state energy") has leading term $-\lambda n(n-1)/2$ as can be seen from Eq. (1.7). Inspired by this, we then propose a naive form for $F_n(L)$,

$$F_n(L)=[nL-w(nL^{1/3})]\frac{\lambda^2}{6}g[(n+1)\lambda]\ , \tag{4.3}$$

where g is a function for which $g(u)\sim3/u$ as $u\to\infty$ but $g(0)=1$. Since we have the exact solution for two replicas, taking $L\to\infty$ with $n=2$ we have

TABLE II. Results for $n=2$ and 3, combinatoric case. The four columns correspond to the first four columns of Table I.

2^3q_c ($n=3$)	r	2^2q_c ($n=2$)	E_3/E_2
0.050	74.184 680	0.279 652	2.351 051
0.150	21.187 827	0.457 778	2.427 939
0.300	8.288 775	0.621 980	2.535 493
0.450	4.208 525	0.741 484	2.669 682
0.650	1.868 268	0.862 842	2.920 099
0.800	0.945 984	0.932 702	3.202 899
0.950	0.301 962	0.986 178	3.685 278
0.965	0.238 234	0.990 570	3.760 371
0.980	0.168 209	0.994 772	3.847 120

$$L\ln4q_2=[2L-2^3L]g(3\lambda)\lambda^2/6\ ,$$

where from the first line of Eq. (2.18) $4q_2=(1+2r)/(1+r)^2=(2e^{\lambda}-1)e^{-2\lambda}$, and therefore

$$g(u)=[3u-9\ln(2-e^{-u/3})]/u^2\ . \tag{4.4}$$

We then denote $[n-n^3]\lambda^2g[(n+1)\lambda]/6$ by $E_n(\text{pred})$ so that $E_2(\text{pred})=E_2$ exactly. We find then that $E_3(\text{pred})$ works pretty well in the intermediate-λ regime (see Table I). For small λ it is not quite as good as the Bethe ansatz.

For the combinatoric case, at very large λ, the particles will branch out and then come back soon; hence, the leading term of "the growing rate" will be less than $-\lambda n(n-1)/2$. We have not found an analog of (4.3) for this case.

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APPENDIX

The free propagator $G_q^k(y_1,y_2,y_3)$ satisfies the discrete Helmholtz equation

$$\begin{aligned}G_q^k(y_1,y_2,y_3)=\delta_{y_1,0}\delta_{y_2,0}\delta_{y_3,0}+q\{&2G_q^k(y_1,y_2,y_3)\cos3k+e^{ik}[G_q^k(y_1+1,y_2-1,y_3)\\&+G_q^k(y_1-1,y_2,y_3+1)+G_q^k(y_1,y_2+1,y_3-1)]\\&+e^{-ik}[G_q^k(y_1-1,y_2+1,y_3)+G_q^k(y_1+1,y_2,y_3-1)+G_q^k(y_1,y_2-1,y_3+1)]\}\ .\end{aligned} \tag{A1}$$

Since we are calculating for $k=0$, we may drop the superscript and have

$$\begin{aligned}(1-2q)G_q(y_1,y_2,y_3)=&\delta_{y_1,0}\delta_{y_2,0}\delta_{y_3,0}\\&+q\sum_{\text{NN}}G_q(y_1',y_2',y_3')\ ,\end{aligned} \tag{A2}$$

where the sum is over the "six nearest neighbors (NN)" on the triangular lattice. (cf. Eq. (2.1) of [15].)

To calculate $G_q^{(k=0)}$ numerically, we adapt the method of [15] where the corresponding function is found on a square lattice. This method is based on the following two observations.

(1) G_q satisfies not only (A2) but also a second exact equation which is the discrete analog of the continuum statement that ∇G_q is radially directed. This "gradient equation" (2.7) in [15], can be stated for either lattice as

$$\sum_{\mu}\mu G_q(\mathbf{r}+\mu)\|\mathbf{r}\ , \tag{A3}$$

where $\mathbf{r}$ is the position vector of a lattice point, and $\mathbf{r}+\mu$ ranges over the (4 or 6) neighbors of that point. (In two

dimensions the statement that two vectors are parallel boils down to a single equation relating their components.)

(2) If the values of G_q are known on a suitable strip extending from the origin to infinity, then they can be extended to a wedge spreading out to a certain angle θ on each side of the strip, by the use of (A3) *alone* [not (A2)]; and this extension (up to angle θ) does not magnify the initial rounding error. For the square lattice the strip must be 2 rows wide and along one of the major axes (e.g., $y_2=0,1$); the angle θ is 45°. For the triangular lattice we find that the strip must be 4 rows wide (e.g., $y_3-y_2=0,1,2,3$), and the angle θ is 30°. In both lattices θ turns out to be just enough so that the wedge accounts by symmetry for the whole plane. Proposition (1) can be derived analytically in the same way as in [15]; proposition (2) is empirical, but unmistakably true. The extension from strip to wedge is extremely rapid since the application of (A3) requires only a few rational operations. Thus, for the point ($y_1=-6$, $y_2=1$, $y_3=5$) which lies just outside the strip ($y_3-y_2>3$) but inside the wedge ($y_2\geq 0$) (see Fig. 1), (A3) applied to $\mathbf{r}=(-6,2,4)$ yields $G_q(-6,1,5)$ by the following equation:

$$G_q(-6,1,5)=G_q(-6,3,3)+\tfrac{2}{3}[G_q(-7,3,4)-G_q(-5,1,4)]-\tfrac{1}{3}[G_q(-7,2,5)-G_q(-5,2,3)]\ . \qquad \text{(A4)}$$

The less trivial part of the computation is the evaluation of G_q on the strip $0\leq y_3\leq y_2\leq 3$. The idea is that by combining (A2) with (A3) one obtains self-contained linear equations involving *only* those values of G_q for points in the strip. Taking starting values for G_q near the origin, one can iterate these equations along the strip arbitrarily far. But these "strip equations" admit extra solutions having nothing to do with G_q: lattice artifacts, some of which grow rapidly. This embarrassment is turned to positive use by a procedure which "fine tunes" the starting values of G_q so as to eliminate the growing solution.

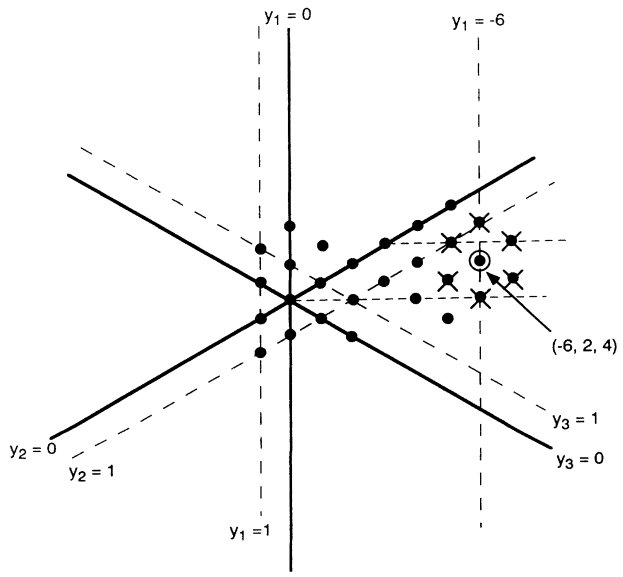

FIG. 1. The triangular lattice has a threefold rotational symmetry. The wedge is in the $y_1\leq 0$ region between line $y_3=y_2$ and line $y_2=0$ inclusive. The strip used to generate the whole wedge is marked by short dashed lines. The point $(-6,2,4)$ is circled and the points around it are marked by $\times$ [cf. Eq. (A4), and text therein].

In both the square and triangular lattice, when symmetry is taken into account, one needs three independent values of G_q: in the square lattice $G_q(0,0)$, $G_q(1,0)=G_1(0,1)$, and $G_q(1,1)$; and in the triangular lattice $G_q(0,0,0)$, $G_q(-1,0,1)=G_q(0,-1,1)$, and $G_q(-1,-1,2)$. The self-contained strip equations admit four solutions of which two can be recovered in the continuum and two are lattice artifacts; of the latter, one grows rapidly and the other dies away. Only *three combinations are consistent with symmetry near the origin.* Thus, by varying the starting values one gets arbitrary combinations of the following three solutions.

(1) A lattice artifact growing by a factor of ~ 6 at each iteration.

(2) A homogeneous solution analogous to $I_0(Mr)$, growing roughly as e^{Mr} where (on our strip), $r=|y_1|$ and M is fixed in accordance with (2.2) of [15], so that for us $M^2=(1/q)-8$.

(3) The inhomogeneous solution that we are looking for, analogous to $K_0(Mr)$ and decaying as e^{-Mr}.

Only the free case ($q=\frac{1}{8}$), which corresponds to the "massless" case of $M=0$, has been discussed in detail in [15]. In that limit the solutions (2) and (3) can be resolved into a logarithmic part and a constant part. The latter can be eliminated by rewriting both (A2) and (A3) in terms of *differences* of G_q from one point to another. Thus, one only has two starting parameters. One starting parameter is determined by the size of the source term in (A2). The other is repeatedly fine tuned so as to eliminate the rapidly growing solution.

When q is less than $\frac{1}{8}$, we have three starting parameters and three solutions, of which two must be eliminated. The algorithm required for this is of the same complexity as for the "massless" case in three dimensions. A program incorporating such an algorithm was written and used for another computation [16], but the algorithm has not been described in detail. Since its design involved some difficulty, we shall describe it in what follows.

For each integer $n\geq 0$, we define a column vector $\chi(n)$ as

$$\chi(n)=\begin{bmatrix} G_q(-2n,n,n) \\ G_q(-2n,n-1,n+1) \\ G_q(-2n-1,n,n+1) \\ G_q(-2n-1,n-1,n+2) \end{bmatrix}\ . \qquad \text{(A5)}$$

Then by applying (A3) at $(-2n-1,n,n+1)$ and at $(-2n-2,n,+2)$, and (A2) at these points and at $(-2n-2,n+1,n+1)$, we obtain five equations from which $G_q(-2n-2,n-1,n+3)$ can be eliminated, leaving four equations which can be written as

$$\chi(n+1)=M(n)\chi(n)\ , \qquad \text{(A6)}$$

where M is a 4×4 matrix whose elements are rational in n. (A 4×4 matrix is also obtained on the square lattice.)

The three independent solutions of (A6) possessing the required symmetry of the origin may be called χ_A,χ_B,χ_C and can be obtained by iterating (A6) from

$$\chi_A(0)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad \chi_B(0)=\begin{bmatrix}6q\\1-2q\\1-2q\\0\end{bmatrix},\quad \text{and } \chi_C(0)=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}. \tag{A7}$$

Applying (A2) at the origin, and assuming symmetry, we see that the desired χ must be of the form

$$\chi(n)=\frac{1}{1-2q}[\chi_C(n)-\alpha\chi_A(n)-\beta\chi_B(n)]\ , \tag{A8}$$

where α and β are to be chosen to obtain good behavior at $n\to\infty$.

We anticipate that, in general, the two growing eigenvalues will be unequal, and after many iterations both χ_A and χ_B will be dominated by the larger one. But then the determination of α and β, from simultaneous linear equations requiring two components of χ to remain small, will be a small difference of large terms. To prevent this, we shall make a "preliminary replacement" of χ_B by

$$\chi_{B'}(n)=\chi_B(n)-\gamma\chi_A(n)\ , \tag{A9}$$

where γ will be chosen to eliminate the larger eigenvalue from $\chi_{B'}$. We shall then have

$$\chi(n)=\frac{1}{1-2q}[\chi_C(n)-\alpha'\chi_A(n)-\beta\chi_{B'}(n)]\ , \tag{A10}$$

where $\alpha'=\alpha-\beta\gamma$; and the determination of α' and β from simultaneous equations will not be imprecise, because χ_A and $\chi_{B'}$ will not be quasiparallel vectors.

We proceed by first developing χ_A. Since the floating-point exponent is limited by hardware, we must periodically rescale. We have obtained better results with shorter periods, for example rescaling when the components of χ_A have reached $\sim10^{15}$, although our machine allows $\sim10^{25}$. Thus, in region 0 ($0\le n<n_1$) we are recording χ_A; in region 1 ($n_1\le n<n_2$) we are actually storing $\epsilon_1^A\chi_A$ where $\epsilon_1^A\sim10^{-15}$; in region 2, $\epsilon_2^A\chi_A$ where $\epsilon_2^A\sim10^{-30}$; and so on. The quantities stored range in magnitude from ~1 to $\sim10^{15}$.

We then develop χ_B in the same manner, stopping at the end of each region to replace it by $\chi_{B'}$. This is done as follows. At the end of region 0 (the regions have been defined once and for all during the calculation of χ_A) we compute a number γ_0 as $\chi_B^{(i)}(n_1)/\chi_A^{(i)}(n_1)$ where the superscript (i) designates a particular component. Then we define

$$\chi_{B_1}(n)\equiv\chi_B(n)-\gamma_0\chi_A(n) \tag{A11}$$

for all n. Since χ_A asymptotically does not have the designated component much smaller than the others—in principle one should verify this—χ_{B_1} is not quasiparallel to $\chi_A(n_1)$.

To obtain χ_{B_1} accurately, we apply (A11) not at $n=n_1$ but at $n=0$, and then reiterate (A6) through region 0 to obtain χ_{B_1}, discarding χ_B. At n_1 we find that the designated component of χ_B is actually of order 10^{-2} [10^{15} times the smallest precision ($\sim10^{-17}$) of the machine], and the other components are much greater than 1, of an order determined by the "second" growing eigenvalue. We rescale χ_{B_1} by ϵ_1^B to bring these components to $O(1)$, and continue iterating $\epsilon_1^B\chi_{B_1}$ through region 1. At n_2 we find that the components of $\epsilon_1^B\chi_{B_1}$ have reached 10^{15} and we define

$$\epsilon_1^B\chi_{B_2}=\epsilon_1^B\chi_{B_1}-\gamma_1\epsilon_1^A\chi_A$$

with

$$\gamma_1=\epsilon_1^B\chi_{B_1}^{(i)}(n_2)/\epsilon_1^A\chi_A^{(i)}(n_2)\ . \tag{A12}$$

We do not compute χ_{B_2} by iterating (A6) from $n=0$, because the machine rounding error would be magnified according to the first eigenvalue. Instead, we apply (A12) directly to all $n\le n_1$ and iterate (A6) forward from n_1. Again, we rescale at n_2 and stop at n_3 to compute $\gamma_2=\epsilon_2^B\chi_{B_2}^{(i)}(n_3)/\epsilon_2^A\chi_A^{(i)}(n_3)$.

In general, after iterating (A6) for the first time through region j, one stops at n_{j+1}, finds that $\epsilon_j^B\chi_{B_j}(n_{j+1})\sim10^{15}$, and performs the following procedures.

(1) Compute $\gamma_j=\epsilon_j^B\chi_{B_j}^{(i)}(n_{j+1})/\epsilon_j^A\chi_A^{(i)}(n_{j+1})$.

(2) Define $\chi_{B_{j+1}}$ by

$$\epsilon_j^B\chi_{B_{j+1}}(n)=\epsilon_j^B\chi_{B_j}(n)-\gamma_j\epsilon_j^A\chi_A(n)\ . \tag{A13}$$

(3) Compute $\epsilon_{j-1}^B\chi_{B_{j+1}}(n)$ for $n=n_j$ and work backwards through region $j-1$ using (A13); discard the old value $\epsilon_{j-1}^B\chi_{B_j}$. Then work backwards through region $j-2$, and continue to the origin if necessary. However, the program may be told to stop when it reaches a region j' for which $(\epsilon_{j'}^B/\epsilon_j^B)\gamma_j(\epsilon_j^A/\epsilon_{j'}^A)\chi_A(n_{j'+1})$ is less than machine precision, since for $n\le n_{j'}$ we may take χ_{B_j} as the final $\chi_{B'}$.

(4) Iterate (A6) forward from n_j to develop $\epsilon_j^B\chi_{B_{j+1}}$ in region j, discarding the old value $\epsilon_j^B\chi_{B_j}$. At n_{j+1} define an appropriate ϵ_{j+1}^B.

(5) Rescale and continue iterating (A6) to develop $\epsilon_{j+1}^B\chi_{B_{j+1}}$ in region $j+1$. At n_{j+2} stop and repeat from step (1) with $j\to j+1$.

In this way each region is covered twice using (A6) and as many times as needed using (A13). The numbers γ_j are of $O[1]$. The number γ in (A9) is theoretically given by

$$\gamma=\gamma_0+\gamma_1\frac{\epsilon_1^B}{\epsilon_1^A}+\gamma_2\frac{\epsilon_2^B}{\epsilon_2^A}+\cdots\ , \tag{A14}$$

but we do not compute it explicitly since the direct application of (A9) for large n would involve unacceptable errors. The algorithm given above computes $\chi_{B'}=\lim_{j\to\infty}\chi_{B_j}$ to machine accuracy for arbitrarily large

n, as though γ had been determined to whatever precision this would have required if (A9) were used.

Having computed and stored $\chi_{B'}$ by the above method, we turn to χ, which will be developed by a similar series of approximations starting with χ_C. The organization is similar but no rescaling will be needed since χ does not increase indefinitely. The progression $\chi_B \to \chi_{B_1} \to \cdots \chi_{B'}$ is replaced by a more complicated progression, $\chi_C \to \chi_{\tilde{C}} \to \chi_{C_1} \to \chi_{\tilde{C}_1} \to \cdots \chi$, as follows.

Step (1), after iterating (A6) through region j to obtain $\chi_{C_j}(n_{n+1})$, is to compute $\Delta_j = \chi^{(i)}_{C_j}(n_{j+1})/\epsilon_j^A \chi^{(i)}_A(n_{j+1})$. Step (2) is to define

$$\chi_{\tilde{C}}(n) = \chi_{C_j}(n) - \Delta_j \epsilon_j^A \chi_A(n) \ . \tag{A15}$$

Steps (3) and (4) are the same as before to compute with (A15) $\chi_{\tilde{C}_j}$ through region j. But one does not continue to step (5), because $\chi_{\tilde{C}_j}$ is still growing in region j according to the second eigenvalue. Instead, one returns to the following: (1′) Compute,

$$\alpha_j \equiv \frac{\mu_C \nu_B - \mu_B \nu_C}{\mu_A \nu_B - \mu_B \nu_A}, \quad \beta_j \equiv \frac{\mu_A \nu_C - \mu_C \nu_A}{\mu_A \nu_B - \mu_B \nu_A}$$

where (with $\epsilon_j^C \equiv 1$ and $i \neq i'$)

$$\mu_{A,B,C} = \epsilon_j^{A,B,C} \chi^{(i)}_{A,B',\tilde{C}_j}(n_{j+1}) \ ,$$

$$\nu_{A,B,C} = \epsilon_j^{A,B,C} \chi^{(i)}_{A,B',\tilde{C}_j}(n_{j+1}) \ .$$

(2′) Define

$$\chi C_{j+1}(n) = \chi_{\tilde{C}_j}(n) - \alpha_j \epsilon_j^A \chi_A(n) - \beta_j \epsilon_j^B \chi_{B'}(n) \ . \tag{A16}$$

This makes both $\chi^{(i)}_{C_{j+1}}(n_{j+1})$ and $\chi^{(i')}_{C_{j+1}}(n_{j+1})$ small compared to other components; since presumably no linear combination of the two growing solutions has this property, they are both filtered out. One then carries out steps (3) [with (A16)], (4) and (5) to obtain $\chi_{C_{j+1}}$ through region $j+1$, and repeats from (1) with $j \to j+1$.

Thus, in obtaining χ one covers each region three times with (A6). The extra iteration is needed because if χ_{C_j} were used directly in (1′) and (2′) instead of $\chi_{\tilde{C}_j}$, the precision in β_j would be insufficient to filter out the second growing solution.

Despite its complexity, this algorithm is extremely rapid in execution and amazingly accurate. The strip-and-wedge computation to obtain G_q for all sites within 100 steps of the origin consumes less than a second (VAX computer CPU), and the resulting numbers satisfy (A2) throughout the plane to $\sim 10^{-16}$ absolute accuracy (machine precision $\sim 10^{-17}$).

[1] M. Kardar, Nucl. Phys. B **290**, 582 (1987).
[2] E. Medina and M. Kardar, Phys. Rev. Lett. **66**, 3187 (1991).
[3] Marc Mézard, J. Phys. (Paris) **51**, 1831 (1990).
[4] J. M. Kim, M. A. Moore, and A. J. Bray, Phys. Rev. A **44**, 2345 (1991).
[5] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[6] M. Eden (unpublished).
[7] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A **34**, 5091 (1986).
[8] D. A. Huse and C. L. Henley, Phys. Rev. Lett. **54**, 2708 (1985).
[9] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[10] J. M. Burgers, *The Nonlinear Differential Equation* (Reidel, Boston, 1974).
[11] G. Parisi, J. Phys. (Paris) **51**, 1595 1990).
[12] Since completing this work we have learned that a form equivalent to (1.8) has also been introduced by Medina *et al.*, J. Stat. Phys. **71**, 967 (1993).
[13] J. Krug, P. Meakin, and T. Halpin-Healy, Phys. Rev. A **45**, 638 (1992).
[14] E. Medina and M. Kardar, Phys. Rev. B **46**, 9984 (1992).
[15] R. Friedberg and O. Martin, J. Phys. A **20**, 5095 (1987).
[16] R. Friedberg and O. Martin, J. Phys. A **23**, 1575 (1990).